

ON THE DIMENSIONAL REDUCED THEORIES

G. COGNOLA

*Dipartimento di Fisica, via Sommarive, Povo,
Trento , Italy*

E-mail: cognola@science.unitn.it

S. ZERBINI

*Dipartimento di Fisica, via Sommarive, Povo,
Trento , Italy*

E-mail: ZERBINI@science.unitn.it

The procedure of the dimensional reduction related to the partition function of a quantum scalar field living in curved space-time which is the warp product of symmetric space is investigated.

In this contribution, we would like to revisited the issue related to dimensional reduction, and this is used when one is dealing with quantum fields living on space-times having some symmetries, namely the D-dimensional space-time is the "warp" product $M_P \times \Sigma_Q$, where Σ_Q is a Q-dimensional symmetric space with constant curvature. Such investigation is mainly motivated by recent approaches to black holes physics, initiated in ¹ and continued in ^{2,3} and the calculation of the effective action after and before the dimensional reduction ⁴.

The idea is very simple: since a generic black hole has a large symmetry in the horizon sector, one may consider the two dimensional related reduced theory for which the effective action may be obtained functionally integrating the corresponding conformal anomaly. This procedure gives rise the problem of the validity of the approximation and this will be discussed here. A related issue is the so called dimensional-reduction anomaly ^{5,6}.

Let us consider a scalar field Φ propagating in the above mentioned space.

$$L_D = -\Delta_D + m^2 + \xi R_D, \quad (1)$$

in which m^2 is a possible mass term and ξR_D a suitable "potential term", describing the non-minimal coupling with the gravitational field. The "exact" theory, namely the non-dimensional reduced one, may be described by the path integral (Euclidean partion function)

$$Z = \int D\Phi e^{-\int dV_D \Phi L_D \Phi} = e^{-\Gamma}. \quad (2)$$

The effective action Γ has to be regularised and may be expressed by means

of a zeta-regularised functional determinant^{7,8,9} (for recent reviews, see^{10,11})

$$\Gamma = -\ln Z = -\frac{1}{2} [\zeta'(0|L_D) + \ln \mu^2 \zeta(0|L_D)] , \quad (3)$$

μ^2 being the re normalization parameter. Here, the zeta-function is defined by means

$$\zeta(s|L_D) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} K_t, \quad K_t = \text{Tr} e^{-tL_D}, \quad (4)$$

valid for $\text{Re } s > D/2$. Here $\text{Tr} e^{-tL_D} = \sum_i e^{-t\lambda_i}$, λ_i being the eigenvalues of L .

One may use other regularisation procedures. As an example, the dimensional regularisation is defined by

$$\begin{aligned} \Gamma_\varepsilon &= -\frac{1}{2} \int_0^\infty dt t^{\varepsilon-1} \text{Tr} e^{-tL_D} = -\frac{1}{2} \Gamma(\varepsilon) \zeta(\varepsilon|L_D) \\ &= -\frac{1}{2} \left(\frac{\zeta(0|L_D)}{\varepsilon} + \zeta'(0|L_D) + \gamma \zeta(0|L_D) + O(\varepsilon) \right). \end{aligned} \quad (5)$$

Other regularizations may be used with t^ε substituted by a suitable regularisation function $g_\varepsilon(t)$ (see, for example¹²). Recall that the zeta-function regularisation is a *finite* regularisation and corresponds to the choice

$$g_\varepsilon(t) = \frac{d}{d\varepsilon} \left(\frac{t^\varepsilon}{\Gamma(\varepsilon)} \right). \quad (6)$$

The other ones, as is clear from Eq. (5), give the same finite part, modulo a re normalization, and contain divergent terms as the cutoff parameter $\varepsilon \rightarrow 0$ and these divergent terms have to be removed by related counter-terms.

As a consequence, as will be shown, a crucial role is played by the quantity $\text{Tr} e^{-tL_D}$. With regard to this quantity, its short- t asymptotics has been extensively studied. For a second-order operator on a boundary-less D -dimensional (smooth) manifold, it reads

$$K_t \simeq \sum_{j=0}^{\infty} A_j(L_D) t^{j-D/2}, \quad (7)$$

in which $A_j(L_D)$ are the Seeley-DeWitt coefficients, which can be computed with different techniques^{13,14}. The divergent terms appearing in a generic regularisation depend on $A_j(L_D)$.

In the sequel, we also shall deal with local quantities, which can be defined by the local zeta-function. With this regard, it is relevant the local short t heat-kernel asymptotics, which reads

$$K_t(L_D)(x) = e^{-tL_D}(x) \simeq \frac{1}{(4\pi)^{D/2}} \sum_{j=0}^{\infty} a_j(x|L_D) t^{j-D/2}, \quad (8)$$

where $a_j(x|L_D)$ are the local Seeley-DeWitt coefficients. The first ones are well known and read

$$\begin{aligned} a_0(x|L_D) &= 1, & a_1(x|L_D) &= \left(-\xi R + -m^2 + \frac{R}{6} \right). \\ a_2(x|L_D) &= \frac{1}{2}(a_1(x|L_D))^2 + \frac{1}{6}\Delta_D a_1(x|L_D) + c_2(x), \end{aligned} \quad (9)$$

where

$$c_2(x) = \frac{1}{180} (\Delta_D R + R^{ijk r} R_{ijk r} - R^{ij} R_{ij}). \quad (10)$$

It may be convenient to re-sum partially this asymptotic expansion and one has¹⁵

$$e^{-tL_D}(x) \simeq \frac{e^{ta_1(x|L_D)}}{(4\pi)^{D/2}} \sum_{j=0}^{\infty} b_j(x|L_D) t^{j-\frac{D}{2}}. \quad (11)$$

The advantage of the latter expansion with respect to the previous one, is due to the fact that now the expansion b_j coefficients depend on the potential only through its derivatives. One has

$$\begin{aligned} b_0(x|L_D) &= 1, & b_1(x|L_D) &= 0, \\ b_2(x|L_D) &= -\frac{1}{6}\Delta_D V + \frac{1}{36}\Delta_D R + c_2(x). \end{aligned} \quad (12)$$

Since the exact expression of the local zeta-function is known only in a limited number of cases, one has to make use of some approximation. If the first coefficient $a_1(x|L_D)$ is very large and negative and this is true if the mass is very large, one may obtain an asymptotics expansion of the local zeta-function by means of the short t expansion (8) and the Mellin transform, namely¹²

$$\begin{aligned} \zeta(s|L_D)(x) &\simeq \frac{\Gamma(s - \frac{D}{2})}{(4\pi)^{\frac{D}{2}} \Gamma(s)} (-a_1(x|L_D))^{\frac{D}{2}-s} \\ &+ \sum_{j=2}^{\infty} \frac{\Gamma(s + j - \frac{D}{2})}{(4\pi)^{\frac{D}{2}} \Gamma(s)} (-a_1(x|L_D))^{\frac{D}{2}-s-j} b_j(x|L_D). \end{aligned} \quad (13)$$

The latter expansion directly gives also the analytic continuation in the whole complex plane. The global zeta-function can be obtained integrating over the manifold.

Now, let us introduce the dimensional-reduced theory according to^{1,5}. We indicate by $\tilde{\mathcal{M}}^D$ a D -dimensional Riemannian manifold with metric $\tilde{g}_{\mu\nu}$ and coordinates \tilde{x}^μ ($\mu, \nu = 1, \dots, D$) and by \mathcal{M}^P and $\hat{\mathcal{M}}^Q$ ($Q = D - P$) two sub-manifolds with coordinates x^i ($i, j = 1, \dots, P$) and \hat{x}^a ($a, b = P + 1, \dots, D$) and metrics g_{ij} and \hat{g}_{ab} respectively, related to $\tilde{g}_{\mu\nu}$ by the warped product

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = g_{ij}(x) dx^i dx^j + e^{-2\sigma(x)} \hat{g}_{ab}(\hat{x}) d\hat{x}^a d\hat{x}^b. \quad (14)$$

Here, $\hat{\mathcal{M}}^Q = \Sigma_Q$ is a constant curvature symmetric space.

We shall use the notation $\tilde{R}_{\beta\gamma\delta}^\alpha$, R_{jmn}^i and \hat{R}_{bcd}^a for Riemann tensors in $\tilde{\mathcal{M}}^D$, \mathcal{M}^P and $\hat{\mathcal{M}}^Q$ respectively, and similarly for all other quantities. In the Appendix A, one can find the relationship between the geometrical quantities related to the sub-manifolds.

We start with a scalar field $\Phi(\tilde{x})$ in the Riemannian manifold $\tilde{\mathcal{M}}^D$. The Laplacian-like operator reads

$$L_D \Phi(\tilde{x}) = \tilde{L} \Phi(\tilde{x}) = (-\tilde{\Delta} + \xi \tilde{R} + m^2) \Phi(\tilde{x}) = (L + e^{2\sigma} \hat{L}) \Phi(\tilde{x}), \quad (15)$$

where

$$L = -\Delta + Q\sigma^k \nabla_k + \xi [R + 2Q\Delta\sigma - Q(Q+1)\sigma^k \sigma_k] + m^2, \quad (16)$$

$$\hat{L} = -\hat{\Delta}. \quad (17)$$

In order to dimensionally reduce the theory, let us introduce the harmonic analysis on Σ_Q by means of

$$\hat{L} Y_\alpha(\hat{x}) = \lambda_\alpha Y_\alpha(\hat{x}), \quad (18)$$

λ_α, Y_α being the eigenvalues and eigenfunctions of \hat{L} on the symmetric space $\Sigma_Q = \hat{\mathcal{M}}^Q$. For any scalar field in $\tilde{\mathcal{M}}^D$, we can write

$$\Phi(\tilde{x}) = \sum_\alpha \phi_\alpha(x) Y_\alpha(\hat{x}) \quad (19)$$

and for the partition function, after integration over Y_α in the classical action,

$$Z^* = \int d[\tilde{\phi}] e^{-\int \tilde{\phi} \tilde{L} \tilde{\phi} \sqrt{\tilde{g}} d^D \tilde{x}} = \prod_\alpha Z_\alpha, \quad (20)$$

where

$$Z_\alpha = \int d[\bar{\phi}_\alpha] e^{-\int \bar{\phi}_\alpha L_\alpha \bar{\phi}_\alpha d^P x} . \quad (21)$$

Here $\bar{\phi} = \sqrt[4]{g}\phi$ and $\bar{\phi}_\alpha = \sqrt[4]{g}\phi_\alpha$ are scalar densities of weight $-1/2$ and the dimensional reduced operators read

$$\begin{aligned} L_\alpha &= -\Delta + V + e^{2\sigma} \lambda_\alpha , \\ V &= m^2 + \xi [R + 2Q\Delta \sigma - Q(Q+1)\sigma^k \sigma_k] - \frac{Q}{2}\Delta \sigma + \frac{Q^2}{4}\sigma^k \sigma_k . \end{aligned} \quad (22)$$

In the following, we will denote by an asterisk all the quantities associated with the dimensional reduced operators. As a result, we formally have

$$Z^* = \prod_\alpha \left(\det \frac{L_\alpha}{\mu^2} \right)^{-1/2} . \quad (23)$$

If we *ignore* the multiplicative anomaly associated with functional determinants, namely the fact that $\ln \det AB \neq \ln \det A + \ln \det B$ for regularized functional determinants¹⁶, we have

$$\Gamma^* = -\ln Z^* = \frac{1}{2} \sum_\alpha \ln \det \frac{L_\alpha}{\mu^2} . \quad (24)$$

This formal expression may be regularised and renormalized and we have

$$\Gamma_\varepsilon^* = -\frac{\mu^{2\varepsilon}}{2} \sum_\alpha \int_0^\infty dt t^{-1} g_\varepsilon(t) \text{Tr} e^{-tL_\alpha} . \quad (25)$$

Removing the cutoff and, for example making use of a finite regularisation, one arrives at

$$\Gamma^* = \frac{1}{2} \sum_\alpha \zeta'(0| \frac{L_\alpha}{\mu^2}) . \quad (26)$$

Within this procedure, a quite natural definition of the dimensional-reduction anomaly is⁵

$$A_{DRA} = \Gamma - \Gamma^* . \quad (27)$$

However, there exists *another* possible procedure: if we *do not remove* the ultraviolet cutoff ε , we may interchange the harmonic sum and the integral and arrive at

$$\Gamma_\varepsilon^* = -\frac{\mu^{2\varepsilon}}{2} \int_0^\infty dt t^{-1} g_\varepsilon(t) K_t^*, \quad (28)$$

where we have introduced the dimensionally reduced heat-kernel trace

$$K_t^* = \sum_\alpha \text{Tr} e^{-tL_\alpha}. \quad (29)$$

It is clear that within this second procedure, the existence of a non vanishing dimensional reduction anomaly is strictly related to the fact whether the identity

$$K_t^* = K_t \quad (30)$$

holds. In the following the validity of the identity (30) will be discussed.

First, if the whole space-time (its Euclidean version) is a symmetric space, it is quite easy to show that Eq. (30) is true. The reason is that in this case, one has at disposal besides the dimensional reduced one, the total harmonic sum (see, for example ^{17,11}).

In general, we shall restrict ourselves to the class of non-trivial warped space-time already considered and make use of the short t heat-kernel expansion. For the exact theory we have (here $L_D = \tilde{L}$)

$$K_t(\tilde{L}) = \text{Tr} e^{-t\tilde{L}} \sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^\infty \tilde{a}_n(\tilde{x}|\tilde{L}) t^n, \quad (31)$$

with

$$\tilde{a}_1 = a_1 + e^{2\sigma} \hat{a}_1 - \frac{Q}{6} \left[\Delta \sigma - \left(\frac{Q}{2} - 1 \right) \sigma^k \sigma_k \right], \quad (32)$$

$$\tilde{a}_2 = a_2 + e^{4\sigma} \hat{a}_2 + e^{2\sigma} a_1 \hat{a}_1 - \frac{1}{90} \sigma^k \nabla_k R - \frac{1}{45} \dots \quad (33)$$

where all quantities with tilde refers to the whole manifold $\tilde{\mathcal{M}}^D$ and all quantities with hat refers to the sub-manifold $\hat{\mathcal{M}}^Q$.

With regard to the dimensional reduced kernel

$$K_t^*(\tilde{L}) = \sum_\alpha \text{Tr} e^{-tL_\alpha}, \quad (34)$$

where

$$\begin{aligned} L_\alpha &= -\Delta + V + e^{2\sigma} \lambda_\alpha , \\ V &= m^2 + \xi [R + 2Q\Delta \sigma - Q(Q+1)\sigma^k \sigma_k] - \frac{Q}{2}\Delta \sigma + \frac{Q^2}{4}\sigma^k \sigma_k , \end{aligned} \quad (35)$$

the short t expansion can be computed by means of a straightforward (but tedious) computation¹⁸, and the result is

$$K_t^*(\tilde{L}) \sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^{\infty} \tilde{a}_n^*(\tilde{x}|\tilde{L}) t^n . \quad (36)$$

where

$$a_1^*(\tilde{x}|\tilde{L}) = \tilde{a}_1 , \quad (37)$$

$$a_2^*(\tilde{x}|\tilde{L}) = \tilde{a}_2 . \quad (38)$$

As a consequence, one has

$$K_t(\tilde{L}) \simeq \frac{e^{t\tilde{a}_1(\tilde{x})}}{(4\pi t)^{D/2}} \left[1 + b_2(\tilde{x}|\tilde{L})t^2 + b_3(\tilde{x}|\tilde{L})t^3 + \dots \right] , \quad (39)$$

$$K_t^*(\tilde{L}) \simeq \frac{e^{t\tilde{a}_1(\tilde{x}|L_D)}}{(4\pi t)^{D/2}} \left[1 + b_2(\tilde{x}|\tilde{L})t^2 + b_3^*(\tilde{x}|\tilde{L})t^3 + \dots \right] . \quad (40)$$

Thus, it is quite natural to make the conjecture that $b_n^*(\tilde{x}|\tilde{L}) = b_n(\tilde{x}|\tilde{L})$ for every n and Eq. (30) holds exactly. Let us discuss about the consequences of this fact.

After the dimensional reduction, as far as the effective action is concerned, the operation of renormalization (addition of counterterms and remotion of the cutoff) and the evaluation of the harmonic sum *do not* commute. If we keep fixed and non vanishing the regularisation parameter, we may perform the harmonic sum, and if (30) holds, we may reconstruct the exact partition function, after renormalization. In such case, it is evident that no dimensional reduction anomaly occurs.

On the other hand, one may remove the cutoff, adding the necessary counterterms or using a finite regularisation like the zeta-function one and perform the harmonic sum at the end. In this case, as stressed in reference⁵, one

has to correct the result by adding dimensional reduction anomaly terms. The reason of this possible discrepancy has been explained in ⁵ as mainly due to the necessity of the regularisation and renormalization of the effective action in spaces with different dimensions. There, it has also been observed a possible connection with the multiplicative anomaly.

Regarding this issue, there exists also a mathematical reason for the necessity of these reduction anomaly terms. In fact, the harmonic sum of the renormalized dimensionally reduced effective action diverges and the dimensional anomaly reduction terms are also necessary to recover the exact and *finite* result. This fact stems also from the necessity of the presence of the multiplicative anomaly, since it also diverges, being associated with a product of an infinite number of dimensional reduced operators.

It may be convenient to illustrate the dimensional reduction procedure in the simplest example one can deal with, namely a free massive scalar field in the Euclidean version of the D -dimensional Minkowski space-time. We may decompose $R^D = R \times R^{D-1}$, thus $\hat{M} = R^{D-1}$, and $\sigma = 0$, and $L_{\vec{k}} = -\partial_\tau^2 + m^2 + (\vec{k})^2$. It is easy to show that (30) holds, since

$$\text{Tr } e^{-tL_{\vec{k}}} = V(R^{D-1})e^{-tm^2} \frac{e^{-t\vec{k}^2}}{(4\pi t)^{1/2}}. \quad (41)$$

For D odd, the exact regularized partition function is

$$\Gamma = -\frac{V(R^D)\Gamma(-D/2)}{2(4\pi)^{D/2}}m^D. \quad (42)$$

On the other hand, the partial reduced effective actions are

$$\Gamma_{\vec{k}} = -\frac{V(R)\Gamma(-1/2)}{2(4\pi)^{1/2}}(m^2 + \vec{k}^2)^{1/2}. \quad (43)$$

Thus, $\Gamma^* = \sum_{\vec{k}} \Gamma_{\vec{k}}$ is badly divergent. However, it is possible to show that the finite part of this divergent integral reproduces Γ . In this particular case, the dimensional reduction anomaly must cancel the divergent part.

As a consequence, any approximation ² based on the truncation in the harmonic sum of the dimensional reduced theory, may lead, with regard to the comparison with the exact theory, to incorrect conclusions (see also the discussions and further references reported in ³).

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